

# Exponential rate of $L_p$ - convergence of intrinsic martingales in supercritical branching random walks\*

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**SUMMARY.** Let  $W_n, n \in \mathbb{N}_0$  be an intrinsic martingale with almost sure limit  $W$  in a supercritical branching random walk. We provide criteria for the  $L_p$ -convergence of the series  $\sum_{n \geq 0} e^{an}(W - W_n)$  for  $p > 1$  and  $a > 0$ . The result may be viewed as a state-

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\*A.Iksanov, S. Polotsky and U. Rösler were supported by the German Research Foundation (project no. 436UKR 113/93/0-1). The research leading to the present paper has been mainly conducted during visits to University of Kiev (Rösler), to University of Kiel (Iksanov and Polotsky), and to University of Münster (Iksanov). Financial support obtained from these institutions is gratefully acknowledged.

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ment about the exponential rate of convergence of  $\mathbb{E}|W - W_n|^p$  to zero.

MSC: Primary: 60J80, 60F25

*Key words:* supercritical branching random walk; weighted branching process; martingale; random series;  $L_p$ -convergence, Burkholder's inequality

## 1 Introduction and main results

We start by recalling a definition of the branching random walk. Consider a population starting from one ancestor located at the origin and evolving like a Galton-Watson process but with the generalization that individuals may have infinitely many children. All individuals are residing in points on the real line, and the displacements of children relative to their mother are described by a copy of a locally finite point process  $\mathcal{M} = \sum_{i=1}^J \delta_{X_i}$  on  $\mathbb{R}$ , and for different mothers these copies are independent. Note once again that the random variable  $J = \mathcal{M}(\mathbb{R})$  giving the offspring number may be infinite with positive probability. For  $n \in \mathbb{N}_0 := \{0, 1, \dots\}$  let  $\mathcal{M}_n$  be the point process that defines the positions on  $\mathbb{R}$  of the individuals of the  $n$ -th generation. The sequence  $\mathcal{M}_n, n \in \mathbb{N}_0$  is called a *branching random walk (BRW)*. In what follows we always assume that  $\mathbb{E}J > 1$  (supercriticality) which ensures survival of the population with positive probability.

Every BRW is uniquely associated with a *weighted branching process (WBP)* to be formally introduced next: Let  $\mathbf{V} := \bigcup_{n \geq 0} \mathbb{N}^n$  be the infinite Ulam-Harris tree of all finite sequences  $v = v_1 \dots v_n$  with root  $\emptyset$  ( $\mathbb{N}^0 := \{\emptyset\}$ ) and edges connecting each  $v \in \mathbf{V}$  with its successors  $vi, i = 1, 2, \dots$ . The length of  $v$  is denoted as  $|v|$ . Call  $v$  an individual and  $|v|$  its generation number. Associate with every edge  $(v, vi)$  of  $\mathbf{V}$  a nonnegative random variable  $L_i(v)$  (weight) and define recursively  $L_\emptyset := 1$  and  $L_{vi} := L_i(v)L_v$ . The random variable  $L_v$  may be interpreted as the total multiplicative weight assigned to the unique path from the root  $\emptyset$  to  $v$ . For any  $u \in \mathbf{V}$ , put similarly  $L_\emptyset(u) := 1$  and  $L_{vi}(u) := L_i(v)L_v(u)$ . Then  $L_v(u)$  gives the total weight of the path from  $u$  to  $uv$ . Provided that  $L_i(v), v \in \mathbf{V}, i \in \mathbb{N}$  consists of i.i.d. random variables, the pair  $(\mathbf{V}, \mathbf{L})$  with  $\mathbf{L} := (L_v(w)), v \in \mathbf{V}, w \in \mathbf{V}$  is called a WBP with associated BRW  $\mathcal{M}_n, n \in \mathbb{N}_0$  defined as  $\mathcal{M}_n = \sum_{|v|=n} \delta_{\log L_v}(\cdot \cap \mathbb{R})$ . The  $\log L_v > -\infty$  for  $v \in \mathbb{N}^n$  are thus the positions of the individuals alive

in generation  $n$ . Note that, if  $u\mathbf{V} := \{uv : v \in \mathbf{V}\}$  denotes the subtree of  $\mathbf{V}$  rooted at  $u$ , then the WBP on this subtree is given by  $(u\mathbf{V}, \mathbf{L}(u))$ , where  $\mathbf{L}(u) := (L_u(v)), v \in \mathbf{V}$ .

Next define

$$Z_n := \sum_{|v|=n} L_v \quad \text{and} \quad m(r) := \mathbb{E} \sum_{|v|=n} L_v^r$$

for  $n \in \mathbb{N}_0$ ,  $r > 0$  and suppose that  $m(1) < \infty$ . If  $m$  is differentiable at  $r$ , then

$$m'(r) = \mathbb{E} \left( \sum_{|v|=1} L_v^r \log L_v \right). \quad (1)$$

In those cases where the right hand expectation exists but is  $-\infty$  or  $+\infty$  (which can only happen when  $r$  is a left or right endpoint of the possibly degenerate interval  $\{r : m(r) < \infty\}$ ) we take (1) as the definition of  $m'(r)$ .

Let  $\mathcal{F}_0$  be the trivial  $\sigma$ -field,  $\mathcal{F}_n := \sigma(L_i(v) : i \in \mathbb{N}, |v| < n)$  for  $n \in \mathbb{N}$  and  $\mathcal{F}_\infty := \sigma(\mathcal{F}_n : n \in \mathbb{N}_0)$ . The sequence  $(W_n, \mathcal{F}_n), n \in \mathbb{N}_0$ , where

$$W_n := \frac{Z_n}{m^n(1)}, \quad (2)$$

forms a nonnegative martingale with mean one and is thus a.s. convergent to a limiting variable  $W$ , say, satisfying  $\mathbb{E}W \leq 1$ . It has been extensively studied in the literature, but first results were obtained in [11] and [5]. Note that  $\mathbb{P}\{W > 0\} > 0$  if, and only if,  $W_n, n \in \mathbb{N}_0$  is uniformly integrable. An ultimate uniform integrability criterion was given in [1], earlier results can be found in [5], [14], [12] and [10].

Possibly after switching to the WBP  $(\mathbf{V}, (L_v(w)/m^{|v|}(1), v, w \in \mathbf{V}))$  it is no loss of generality to assume throughout that

$$m(1) = 1.$$

We further impose the condition

$$\mathbb{P}\{W_1 = 1\} < 1 \quad (3)$$

which avoids the trivial situation where  $\mathbb{P}\{W_n = 1\}$  for all  $n \in \mathbb{N}$  and hence  $\mathbb{P}\{W = 1\} = 1$ .

Other WBPs appearing in this work are the afore-mentioned  $(u\mathbf{V}, \mathbf{L}(u))$  for any  $u \in \mathbf{V}$  and  $(\mathbf{V}, \mathbf{L}^r)$ , where  $\mathbf{L}^r := (L_v^r(w)), v, w \in \mathbf{V}$ . The counterparts of  $Z_n, W_n$  for these processes are denoted  $Z_n(u), W_n(u)$  and  $Z_n^{(r)}, W_n^{(r)}$ , respectively, so  $Z_n(u) := \sum_{|v|=n} L_v(u)$ ,  $Z_n^{(r)} := \sum_{|v|=n} L_v^r$  and  $W_n^{(r)} := \frac{Z_n^{(r)}}{m^n(r)}$ .

The main results of this paper will provide necessary and sufficient conditions for the  $L_p$ -convergence ( $p > 1$ ) of the series

$$A := \sum_{n \geq 0} e^{an}(W - W_n), \quad (4)$$

for fixed  $a > 0$ . More precisely, we will derive equivalent necessary and sufficient conditions in the simpler case  $p \geq 2$ , while a necessary and a slightly stronger sufficient condition are presented in the surprisingly intriguing case  $1 < p < 2$ . Plainly, our results give information on the rate of convergence of  $\mathbb{E}|W - W_n|^p$  to zero, as  $n \rightarrow \infty$ . It is therefore useful to recall conditions (which can be found in [13, Theorem 2.1], [10, Corollary 5] or [2, Theorem 3.1]) ensuring that this expectation does go to 0 or, equivalently, that the martingale  $\{W_n : n \in \mathbb{N}_0\}$  converges in  $L_p$ .

**Proposition 1.1.** *Suppose (3) and  $p > 1$ . Then the conditions*

$$\mathbb{E}W_1^p < \infty \quad \text{and} \quad m(p) < 1$$

*are necessary and sufficient for*

$$\lim_{n \rightarrow \infty} \mathbb{E}|W - W_n|^p = 0,$$

*and the latter is equivalent to  $\sup_{n \geq 0} \mathbb{E}W_n^p < \infty$  as well as to  $\mathbb{E}W^p \in (0, \infty)$ .*

Now we are ready to formulate our main results.

**Theorem 1.2.** *Suppose (3),  $a > 0$  and  $p \in (1, 2)$ . Then  $A$  converges in  $L_p$  and almost surely if*

$$\mathbb{E}W_1^r < \infty \quad \text{and} \quad e^a m^{1/r}(r) < 1 \quad \text{for some } r \in [p, 2]. \quad (5)$$

*Conversely, the  $L_p$ -convergence of  $A$  implies*

$$\mathbb{E}W_1^p < \infty \quad \text{and} \quad \inf_{r \in [p, 2]} e^a m^{1/r}(r) \leq 1. \quad (6)$$

**Remark 1.3.** In the case where the function  $r \mapsto m^{1/r}(r)$  attains its minimum at some  $\theta < p$ , i.e.  $m(\theta)^{1/\theta} < m^{1/p}(p)$  for some  $1 < \theta < p$ , our analysis will actually show that the  $L_p$ -convergence of  $A$  even implies

$$\mathbb{E}W_1^p < \infty \quad \text{and} \quad e^a m^{1/p}(p) < 1,$$

see Remark 4.1 after the proof of Theorem 1.2. Similarly, if the function  $r \mapsto m^{1/r}(r)$  attains its minimum at some  $\theta \geq 2$ , the  $L_p$ -convergence of  $A$  implies

$$\mathbb{E}W_1^p < \infty \quad \text{and} \quad e^a m^{1/2}(2) < 1.$$

In other words,

$$\inf_{r \in [p, 2]} e^a m^{1/r}(r) = 1$$

in condition (6) is possible only if the last infimum is attained at some  $r \in [p, 2]$ .

**Theorem 1.4.** *Suppose (3),  $a > 0$  and  $p \geq 2$ . Then  $A$  converges in  $L_p$  if, and only if,*

$$\mathbb{E}W_1^p < \infty \quad \text{and} \quad e^a (m^{1/2}(2) \vee m^{1/p}(p)) < 1, \quad (7)$$

*and in this case  $A$  converges also almost surely.*

**Remark 1.5.** Suppose that  $A$  in (4) exists in the sense of convergence in probability and let  $A(v)$  be the corresponding series for the subtree  $v\mathbf{V}$ . The  $A(v)$ ,  $|v| = 1$ , are independent copies of  $A$  and independent of the  $L_v$ ,  $|v| = 1$ . Moreover, the equation

$$A \stackrel{d}{=} e^a \sum_{|v|=1} L_v A(v) + W - 1 \quad (8)$$

holds true (in fact, even with " $=$ " instead of " $\stackrel{d}{=}$ "). Albeit looking like a stochastic fixed point equation it is not, for the  $A(v)$  are *not* independent of the random variable  $W$ .

## 2 Size-biasing and spinal trees

In the following, we will briefly present some required material on size-biasing and spinal trees in connection with BRW. Generally speaking, size-biasing has proved to be a very effective tool from harmonic analysis in the study

of various branching models. Here we restrict ourselves to a rather informal description of those facts that are needed in this article.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  denote the underlying probability space. As  $W_n$ ,  $n \in \mathbb{N}_0$  constitutes a nonnegative mean one martingale, we can uniquely define a new probability measure  $\widehat{\mathbb{P}}$  on  $(\Omega, \mathcal{F}_\infty)$  via the projections

$$d\widehat{\mathbb{P}}|_{\mathcal{F}_n} = W_n d\mathbb{P}|_{\mathcal{F}_n}$$

for all  $n \in \mathbb{N}_0$ .

Fix  $n$  and define a random variable  $\Xi_n$  taking values in  $\mathbf{V}_n := \{v \in \mathbf{V} : |v| = n\}$  such that

$$\widehat{\mathbb{P}}(\Xi_n = v | \mathcal{F}_\infty) = \frac{L_v}{W_n}.$$

Hence  $\Xi_n, n \in \mathbb{N}$ , picks a node in  $\mathbf{V}_n$  in accordance with the size-biased distribution obtained from  $L_v$ ,  $v \in \mathbf{V}_n$ . Let  $(\Xi_0, \dots, \Xi_n)$  denote the vertices visited by the path connecting the root  $\Xi_0 := \emptyset$  with  $\Xi_n$ . It is not difficult to verify that, conditioned upon  $\mathcal{F}_\infty$ , this random vector constitutes a Markov chain on the subtree  $\mathbf{V}_{\leq n} := \{v \in \mathbf{V} : |v| \leq n\}$  with one-step transition probabilities

$$P(v, vi) := \frac{L_i(v)W_k(v)}{W_{k+1}(vi)}, \quad v \in \mathbf{V}_k, \quad vi \in \mathbf{V}_{k+1}.$$

Though suppressed in the notation, it should be noticed that  $P(\cdot, \cdot)$  depends on  $n$  and on  $\mathcal{F}_\infty$ . The thus obtained random line of individuals  $(\Xi_0, \dots, \Xi_n)$  in  $\mathbf{V}_{\leq n}$  is called its *spine*, and the main observation stated in Proposition 2.1 below is that these individuals produce offspring and pick a position in a different way than the other population members.

Define

$$\mathcal{I}_k := \{i \in \mathbb{N} : \Xi_{k-1}i \neq \Xi_k \text{ and } L_i(\Xi_{k-1}) > 0\}$$

to be the random set of labels  $i$  such that  $\Xi_{k-1}i$  is nonspinal offspring in generation  $k$  of the spinal mother  $\Xi_{k-1}$ . Notice that  $\mathcal{I}_k$  may be empty. Define further

$$\mathcal{G}_n := \sigma\left(\left(\Xi_k, L_{\Xi_k}, \mathcal{I}_k\right)_{1 \leq k \leq n}, \sum_{i \in \mathcal{I}_k} \delta_{L_i(\Xi_{k-1})}\right),$$

$\mathbf{S} = \{(v, L_v) : v \in \mathbf{V}\}$  and  $\mathbf{S}_{\leq n} := \{(v, L_v) : |v| < n\}$ . Following our usual convention, we let  $\mathbf{S}_{\leq n}(v)$  denote the shifted counterpart of  $\mathbf{S}_{\leq n} = \mathbf{S}_{\leq n}(\emptyset)$

rooted at  $v$ , more precisely

$$\mathbf{S}_{\leq n}(v) := \{(vw, L_w(v)) : |w| < n\}.$$

The following proposition, of which parts (a)–(d) appear in a similar form in [9], provides all relevant information on the distribution of  $\mathbf{S}_{\leq n}$  and the spine under  $\widehat{\mathbb{P}}$ .

**Proposition 2.1.** *The following assertions hold true under the probability measure  $\widehat{\mathbb{P}}$  for any fixed  $n \in \mathbb{N}$ :*

- (a) *The random vectors  $(\sum_{i \in \mathcal{I}_k} \delta_{L_i(\Xi_{k-1})}, L_{\Xi_k}/L_{\Xi_{k-1}})$ ,  $1 \leq k \leq n$ , are independent and identically distributed with the same distribution as  $(\sum_{i \in \mathcal{I}_1} \delta_{L_i}, L_{\Xi_1})$ .*
- (b) *Conditioned upon  $\mathcal{G}_n$ , the shifted weighted subtrees  $\mathbf{S}_{\leq n-|v|}(v)$ ,  $v \in \bigcup_{k=1}^n \mathcal{I}_k$ , are independent, and  $\widehat{\mathbb{P}}(\mathbf{S}_{\leq n-|v|}(v) \in \cdot | \mathcal{G}_n) \equiv \mathbb{P}(\mathbf{S}_{n-|v|} \in \cdot)$ .*
- (c) *Putting  $\Pi_k := L_{\Xi_k}$  and  $Q_k := \sum_{i \in \mathbb{N}} L_i(\Xi_{k-1})$  for  $k \in \mathbb{N}_0$ , the random vectors  $(\Pi_k/\Pi_{k-1}, Q_k, |\mathcal{I}_k|)$ ,  $1 \leq k \leq n$ , are independent copies of  $(\Pi_1, Q_1, |\mathcal{I}_1|)$ . Moreover,  $\widehat{\mathbb{E}} \log \Pi_1 = m'(1)$  if  $m'(1)$  exists, while  $\widehat{\mathbb{E}} \log \Pi_1$  does not exist, otherwise.*
- (d) *For any nonnegative measurable  $f : [0, \infty) \rightarrow [0, \infty)$*

$$\widehat{\mathbb{E}} f(\Pi_n) = \mathbb{E} \left( \sum_{|v|=n} L_v f(L_v) \right). \quad (9)$$

- (e) *For any nondecreasing and concave function  $f : [0, \infty) \rightarrow [0, \infty)$*

$$\widehat{\mathbb{E}} f(W_n) \leq \widehat{\mathbb{E}} f \left( \sum_{k=0}^{n-1} \Pi_k Q_{k+1} \right). \quad (10)$$

We omit the proof of this result and mention only that parts (a)–(d) follow along similar arguments as those provided for supercritical Galton-Watson trees by Lyons et al. [15]. Equality (9) may also be found in [7]. Part (e) has been derived by Alsmeyer and Iksanov [1], see their argument to derive formula (60).

For any  $\theta \geq 0$  such that  $m(\theta) < \infty$ , the previously defined size-biasing can clearly be done as well with respect to  $W_n^{(\theta)}, n \in \mathbb{N}_0$  by introducing the probability measure  $\widehat{\mathbb{P}}_\theta$  on  $\mathcal{F}_\infty$  defined via the projections

$$d\widehat{\mathbb{P}}_{\mathcal{F}_n}^{(\theta)} = W_n^{(\theta)} d\mathbb{P}_{|\mathcal{F}_n}$$

for  $n \in \mathbb{N}_0$ . Notice that

$$\frac{d\widehat{\mathbb{P}}_{\mathcal{F}_n}^{(\theta)}}{d\widehat{\mathbb{P}}_{|\mathcal{F}_n}} = \frac{\Pi_n^{\theta-1}}{m^n(\theta)} \quad (11)$$

for each  $n \in \mathbb{N}_0$ , because

$$\begin{aligned} \widehat{\mathbb{P}}^{(\theta)}(B) &= \mathbb{E} \left( \sum_{|v|=n} \frac{L_v^\theta}{m^n(\theta)} \mathbf{1}_B \right) \\ &= \mathbb{E} \left( W_n \sum_{|v|=n} \frac{L_v}{W_n} \frac{L_v^{\theta-1}}{m^n(\theta)} \mathbf{1}_B \right) \\ &= \widehat{\mathbb{E}} \left( \sum_{|v|=n} \widehat{\mathbb{P}}(\Xi_n = v | \mathcal{F}_\infty) \frac{L_v^{\theta-1}}{m^n(\theta)} \mathbf{1}_B \right) \\ &= m^{-n}(\theta) \widehat{\mathbb{E}} \left( \Pi_n^{\theta-1} \mathbf{1}_B \right) \end{aligned}$$

for all  $B \in \mathcal{F}_n$ .

### 3 Auxiliary results

The next result will be crucial for our further analysis as explained in the subsequent Remark 3.2.

**Lemma 3.1.** *For any fixed  $a > 0$ , the series  $A$  in (4) converges a.s. (in  $L_p$  for  $p > 1$ ) if, and only if, the same holds true for the series*

$$A' := \sum_{n \geq 0} b_n (W_{n+1} - W_n), \quad (12)$$

where  $b_n := \sum_{k=0}^n e^{ak} = (e^a - 1)^{-1} (e^{a(n+1)} - 1)$  for  $n \in \mathbb{N}_0$ . In this case  $A' = A$  a.s.

*Proof.* Define  $A_m := \sum_{n=0}^m e^{an}(W - W_n)$  for  $m \in \mathbb{N}_0$ . Then

$$\begin{aligned} A_m &= \lim_{l \rightarrow \infty} \sum_{n=0}^m e^{an} \sum_{k=n}^l (W_{k+1} - W_k) \\ &= \lim_{l \rightarrow \infty} \sum_{k=0}^l (W_{k+1} - W_k) \sum_{n=0}^{k \wedge m} e^{an} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} b_{k \wedge m} (W_{k+1} - W_k) \\
&= b_m (W - W_m) + A'_{m-1} \quad \text{a.s.}
\end{aligned} \tag{13}$$

where  $A'_{m-1} := \sum_{k=0}^{m-1} b_k (W_{k+1} - W_k)$ . Now, if  $A$  in (4) converges a.s., then  $\lim_{m \rightarrow \infty} b_m (W - W_m) = 0$  a.s. and thus, by letting  $m$  tend to infinity in (13), we see that  $A'$  converges a.s. and equals  $A$ . Conversely, given the almost sure convergence of  $A'$ , a tail sum analogue of Kronecker's lemma (see [3, Lemma 4.2]) ensures that  $\lim_{n \rightarrow \infty} e^{an} (W - W_n) = 0$  a.s. This in turn allows us to read (13) backwards thus concluding the a.s. convergence of  $A$  as well as  $A = A'$  a.s.

If  $A$  is  $L_p$ -convergent for some  $p > 1$ , then  $\|A_m - A\|_p \rightarrow 0$  and therefore  $e^{am} \|W - W_m\|_p = \|A_{m+1} - A_m\|_p \rightarrow 0$  as  $m \rightarrow \infty$ . Now use (13) to infer with the help of Minkowski's inequality

$$\|A'_{m-1}\|_p \leq b_m \|W - W_m\|_p + \|A_m\|_p$$

and thereupon the  $L_p$ -boundedness of the martingale  $A'_n, n \in \mathbb{N}_0$ . Consequently (see, for example, [16, Proposition IV-2-7] and its proof),  $A'$  defined in (12) converges a.s. as well as in  $L_p$ . Conversely, if  $A'$  is  $L_p$ -convergent, then by an appeal to Burkholder's inequality (see Lemma 3.6 below)

$$\begin{aligned}
b_m^p \mathbb{E} |W - W_m|^p &\leq C b_m^p \mathbb{E} \left( \sum_{n \geq m} (W_{n+1} - W_n)^2 \right)^{p/2} \\
&\leq C \mathbb{E} \left( \sum_{n \geq m} b_n^2 (W_{n+1} - W_n)^2 \right)^{p/2} \\
&\leq C \mathbb{E} |A' - A'_{m-1}|^p \rightarrow 0 \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

where  $C \in (0, \infty)$  is a generic constant that may differ from line to line. With this result we infer from (13)

$$\begin{aligned}
&\|A_{m+n} - A_m\|_p \\
&\leq b_m \|W - W_m\|_p + b_{m+n} \|W - W_{m+n}\|_p + \|A'_{m+n-1} - A'_{m-1}\|_p \\
&\leq 2 \sup_{k \geq m} b_k \|W - W_k\|_p + \|A'_{m+n-1} - A'_{m-1}\|_p \\
&\rightarrow 0 \quad \text{as } m, n \rightarrow \infty
\end{aligned}$$

and thus the asserted  $L_p$ -convergence of  $A$ . □

**Remark 3.2.** (a) As, for each  $n \in \mathbb{N}_0$ ,

$$A'_n = \frac{e^a}{e^a - 1} \sum_{k=0}^n e^{ak} (W_{k+1} - W_k) - \frac{1}{e^a - 1} (W_{n+1} - 1)$$

the proof of Lemma 3.1 may easily be extended to show further that  $A$  converges a.s. (or in  $L_p$  for  $p > 1$ ) if, and only if, this holds true for

$$\hat{A} := \sum_{n \geq 0} e^{an} (W_{n+1} - W_n). \quad (14)$$

In this case,  $\hat{A}$  is readily seen to satisfy

$$\hat{A} \stackrel{d}{=} e^a \sum_{|v|=1} L_v \hat{A}_v + W_1 - 1 \quad (15)$$

with  $\hat{A}_v$  being independent copies of  $\hat{A}$  which are also independent of  $W_1$ . Hence, unlike (8) for  $A$ , (15) constitutes a proper stochastic fixed point equation.

(b) The motivation behind dealing with  $\hat{A}$  in (14) hereafter rather than  $A$  in (4) stems from the fact that the partial sums  $\hat{A}_n := \sum_{k=0}^n e^{ak} (W_{k+1} - W_k)$ ,  $n \in \mathbb{N}_0$ , constitute a martingale whereas those associated with  $A$  do not. This entails that  $\hat{A}$  forms a martingale limit (like  $A'$ ) and as such is easier to deal with. Indeed, as far as the  $L_p$ -convergence ( $p > 1$ ) is concerned, a well-known property of martingales (already used in the previous proof) tells us that it suffices to prove  $\mathbb{E}|\hat{A}|^p < \infty$  or, equivalently,  $L_p$ -boundedness of the  $\hat{A}_n$  (see [16, Proposition IV-2-7]).

The proof of Theorem 1.2 hinges to a large extent on Proposition 3.4 on the functions  $s_n(r)$  defined below. The connection is provided by an application of Burkholder's inequality which in turn is stated for reference as Lemma 3.6 at the end of this section.

**Lemma 3.3.** *Let  $1 < p < 2$  and  $W_n, n \in \mathbb{N}_0$  be uniformly integrable with  $\mathbb{E}W_1^p < \infty$ . Then the function*

$$[1, 2] \ni r \mapsto s_n(r) := \mathbb{E}(Z_n^{(r)})^{p/r} = \mathbb{E}\left(\sum_{|v|=n} L_v^r\right)^{p/r}$$

is decreasing and bounded by  $\sup_{n \geq 0} \mathbb{E}W_n^p$  for each  $n \in \mathbb{N}$ . Furthermore,

$$s_n(r) \begin{cases} \leq s_k(r)s_{n-k}(r), & \text{if } r \in [1, p], \\ \geq s_k(r)s_{n-k}(r), & \text{if } r \in [p, 2] \end{cases} \quad (16)$$

for  $0 \leq k \leq n$ , and

$$\lim_{n \rightarrow \infty} s_n^{1/n}(r) \begin{cases} = \inf_{j \geq 1} s_j^{1/j}(r), & \text{if } r \in [1, p], \\ = \sup_{j \geq 1} s_j^{1/j}(r), & \text{if } r \in [p, 2]. \end{cases} \quad (17)$$

*Proof.* The first assertion follows immediately from  $s_n(1) = \mathbb{E}W_n^p$  and

$$\mathbb{E} \left( \sum_{|v|=n} L_v^r \right)^{p/r} = \mathbb{E} \left( \sum_{|v|=n} L_v^{q \cdot (r/q)} \right)^{p/r} < \mathbb{E} \left( \sum_{|v|=n} L_v^q \right)^{p/q}$$

for any  $1 \leq q < r \leq 2$ , where supercritical branching and strict superadditivity of  $x \mapsto x^{r/q}$  have been utilized. As for (16), we obtain in the case  $r \in [p, 2]$  with the help of Jensen's inequality

$$\begin{aligned} s_n(r) &= \mathbb{E} \left( \sum_{|v|=k} L_v^r Z_{n-k}^{(r)}(v) \right)^{p/r} \\ &= \mathbb{E} \left( (Z_k^{(r)})^{p/r} \left( \sum_{|v|=k} \frac{L_v^r}{Z_k^{(r)}} Z_{n-k}^{(r)}(v) \right)^{p/r} \right) \\ &\geq \mathbb{E} \left( (Z_k^{(r)})^{p/r} \sum_{|v|=k} \frac{L_v^r}{Z_k^{(r)}} (Z_{n-k}^{(r)}(v))^{p/r} \right) \\ &= \mathbb{E} \left( (Z_k^{(r)})^{p/r} \sum_{|v|=k} \frac{L_v^r}{Z_k^{(r)}} \mathbb{E} \left( (Z_{n-k}^{(r)}(v))^{p/r} \middle| \mathcal{F}_k \right) \right) \\ &= \mathbb{E} \left( (Z_k^{(r)})^{p/r} \sum_{|v|=k} \frac{L_v^r}{Z_k^{(r)}} s_{n-k}(r) \right) \\ &= s_k(r) s_{n-k}(r) \end{aligned}$$

for all  $0 \leq k \leq n$ , and this further yields, by superadditivity of  $\log s_n(r)$ , that  $s_n(r)^{1/n}$  converges as  $n \rightarrow \infty$  with limit satisfying (17). If  $r \in [1, p]$  and

thus  $x \mapsto x^{p/r}$  is convex, the above estimation holds with reverse inequality sign.  $\square$

Notice that  $\log s_n(r)$  is always a superadditive or subadditive function but may be infinite. Precise information on the asymptotic value of  $s_n^{1/n}(r)$  as  $n \rightarrow \infty$  is provided by the next lemma. Put  $g(r) := r^{-1} \log m(r)$  with derivative

$$g'(r) = \frac{h(r)}{r^2} \quad \text{with } h(r) := \frac{rm'(r)}{m(r)} - \log m(r) \quad (18)$$

on the interior of  $\mathbb{D} := \{r : m(r) < \infty\}$ . Note that  $[1, p] \subset \mathbb{D}$  if  $W_n, n \in \mathbb{N}_0$  is uniformly integrable and  $\mathbb{E}W_1^p < \infty$ . By supercriticality, the function  $m$  is strictly logconvex which in turn implies that  $h$  is increasing with at most one zero. Therefore, the function  $g$  possesses at most one minimum. Put

$$\vartheta := 2 \wedge \arg \inf_{r \geq 1} g(r) \quad \text{and} \quad \gamma := m^{1/\vartheta}(\vartheta).$$

If  $\vartheta \in \text{int}(\mathbb{D})$  and thus  $m$  is differentiable at  $\vartheta$ , then  $g'(\vartheta) = 0$  may be rewritten as

$$\frac{m'(\vartheta)}{m(\vartheta)} = \frac{1}{\vartheta} \log m(\vartheta). \quad (19)$$

Let us also point out that  $m(r) < 1$  and  $m'(r) < 0$  for all  $r \in (1, \vartheta)$  because  $g(r)$  has negative (right) derivative  $m'(1)$  at 1 as a consequence of uniform integrability of  $W_n, n \in \mathbb{N}_0$ .

**Proposition 3.4.** *Suppose the assumptions of Lemma 3.3 be true and furthermore  $m(p) < 1$ . Let  $\vartheta, \gamma$  be as defined above. Then, if  $p \leq \vartheta$ ,*

$$\lim_{n \rightarrow \infty} s_n^{1/n}(r) = \begin{cases} m^{p/r}(r), & \text{if } r \in [1, \vartheta) \\ \gamma^p, & \text{if } r \in [\vartheta, 2], \end{cases}$$

while, if  $p > \vartheta$ ,

$$\lim_{n \rightarrow \infty} s_n^{1/n}(r) = \begin{cases} m^{p/r}(r), & \text{if } r \in [1, q) \\ m(p), & \text{if } r \in [q, 2], \end{cases}$$

where  $q$  is the unique value in  $(1, \vartheta)$  such that  $g(q) = g(p)$ , i.e.  $m^{1/q}(q) = m^{1/p}(p)$ .

Notice that in both cases above the obtained limit function  $s_\infty(r)$ , say, is continuous at its "critical" value  $\vartheta$ , respectively  $q$ . Also, this limit function for  $p > \vartheta$  converges to the one for  $p = \vartheta$ , for then  $q$  equals  $\vartheta$  as well.

*Proof.* CASE A.  $p \leq \vartheta$  and  $r \in [\vartheta, 2]$ . *Lower estimate*

Since  $s_n(r)$  is decreasing in  $r$  it suffices to show

$$\liminf_{n \rightarrow \infty} s_n^{1/n}(2) \geq \gamma^p. \quad (20)$$

SUBCASE A.1.  $\gamma = m^{1/\vartheta}(\vartheta)$  for  $\vartheta \in (1, 2)$ .

An old result by Biggins [4],[6] tells us that

$$\frac{\log M_n}{n} \rightarrow \frac{1}{\vartheta} \log m(\vartheta) \quad \text{a.s. on } \{W > 0\},$$

where  $M_n := \max_{|v|=n} L_v$ . By using this fact in combination with the obvious inequality

$$(Z_n^{(2)})^{p/2} \geq M_n^p \quad \text{on } \{W > 0\},$$

we infer with the help of Jensen's inequality and Fatou's lemma

$$\begin{aligned} \liminf_{n \rightarrow \infty} s_n^{1/n}(2) &\geq \liminf_{n \rightarrow \infty} \mathbb{E}^{1/n}(M_n^p \mathbf{1}_{\{W > 0\}}) \\ &= \liminf_{n \rightarrow \infty} \mathbb{P}^{1/n}\{W > 0\} \mathbb{E}^{1/n}(M_n^p | W > 0) \\ &\geq \liminf_{n \rightarrow \infty} \mathbb{E}(M_n^{p/n} | W > 0) \\ &= m^{p/\vartheta}(\vartheta) = \gamma^p. \end{aligned}$$

SUBCASE A.2.  $\gamma = m^{1/2}(2)$  (thus  $\vartheta = 2$ ) and  $W_1$  is a.s. bounded.

Then  $m(2) < 1$  and  $m'(2) < 0$  as pointed out after (19). Moreover, the almost sure boundedness  $W_1$  trivially ensures the same for  $W_1^{(2)}$ , in particular  $\mathbb{E}W_1^{(2)} \log^+ W_1^{(2)} < \infty$ . Therefore the mean one martingale  $W_n^{(2)}$ ,  $n \in \mathbb{N}_0$  is uniformly integrable (cf. e.g. [1, Theorem 1.3]) and hence convergent a.s. and in  $L_1$  to a random variable  $W^{(2)}$ . Since  $p/2 < 1$ , it follows that  $\mathbb{E}(W_n^{(2)})^{p/2} \rightarrow \mathbb{E}(W^{(2)})^{p/2}$  and therefore

$$s_n^{1/n}(2) = m^{p/2}(2) \mathbb{E}^{1/n}(W_n^{(2)})^{p/2} \rightarrow m^{p/2}(2) = \gamma^p,$$

as  $n \rightarrow \infty$ . Notice that we have indeed verified the stronger assertion that

$$\lim_{n \rightarrow \infty} \frac{s_n(2)}{m^{pn/2}(2)} = \mathbb{E}(W^{(2)})^{p/2}. \quad (21)$$

SUBCASE A.3.  $\gamma = m^{1/2}(2)$ , general situation.

Here we use a truncation argument. For a constant  $K > 0$  consider the WBP  $(\mathbf{V}, (\bar{L}_v(w), v, w \in \mathbf{V}))$  with

$$\bar{L}_i := L_i \mathbf{1}_{\{L_i \geq 1/K, W_1(v) \leq K\}}, \quad i \in \mathbb{N}, v \in \mathbf{V}. \quad (22)$$

This provides us with a thinning of the original WBP such that  $\bar{m}(\theta) := \mathbb{E}(\sum_{i \geq 1} \bar{L}_i^\theta)$  satisfies

$$\bar{m}(\theta) < \infty \quad \text{and} \quad \bar{m}(\theta) \leq m(\theta)$$

for all  $\theta > 0$ . Moreover, in the obvious notation,

$$\bar{s}_n(\theta) \leq s_n(\theta)$$

for all  $\theta \in [1, 2]$ . Plainly, as  $K \rightarrow \infty$ ,  $\bar{m}$  converges to  $m$  uniformly on compact subsets contained in the interior of  $\mathbb{D}$ . Hence, by choosing  $K$  large enough, we have for the obviously defined  $\bar{\gamma}$  that

$$\bar{\gamma} \geq (1 - \varepsilon)\gamma$$

for any fixed  $\varepsilon \in (0, 1)$ . By applying the result obtained under Subcase A.2 to the normalized WBP  $(\mathbf{V}, (\bar{L}_v(w)/\bar{m}^{|v|}(1), v, w \in \mathbf{V}))$  we now arrive at the desired conclusion here as well.

CASE A.  $p \leq \vartheta$  and  $r \in [\vartheta, 2]$ . *Upper estimate*

The next step is to verify

$$\limsup_{n \rightarrow \infty} s_n^{1/n}(r) \leq \gamma^p \quad (23)$$

for each  $r \in [\vartheta, 2]$  which, in combination with  $s_n(2) \leq s_n(r)$  and (20), clearly gives the assertion of the lemma for  $r \in [\vartheta, 2]$  and  $p \leq \vartheta$ .

Suppose first that  $p < \vartheta$ . Fix any  $\varepsilon > 0$  and  $\theta \in (p, \vartheta)$  such that  $m^{1/\theta}(\theta) \leq (1 + \varepsilon)\gamma < 1$ . Then, by another use of Jensen's inequality,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} s_n^{1/n}(r) &\leq \limsup_{n \rightarrow \infty} s_n^{1/n}(\theta) \\
&= \limsup_{n \rightarrow \infty} \left( m^{pn/\theta}(\theta) \mathbb{E}(W_n^{(\theta)})^{p/\theta} \right)^{1/n} \\
&\leq m^{p/\theta}(\theta) \limsup_{n \rightarrow \infty} \mathbb{E}^{p/n\theta} W_n^{(\theta)} \\
&= m^{p/\theta}(\theta) \\
&\leq (1 + \varepsilon)^p \gamma^p
\end{aligned}$$

which shows (23) as  $\varepsilon > 0$  was picked arbitrarily. Now, if  $p = \vartheta$ , we arrive at the same conclusion by choosing  $\theta = p$  and  $\varepsilon = 0$  in the above estimation.

CASE B.  $p \leq \vartheta$  and  $r \in [1, \vartheta)$ . *Lower estimate*

Here we must verify

$$\liminf_{n \rightarrow \infty} s_n^{1/n}(r) \geq m^{p/r}(r). \quad (24)$$

In view of the truncation (22) described under Subcase A.3 it is no loss of generality to assume directly that  $W_1$  (and thus  $W_1^{(r)}$  as well) is a.s. bounded and  $m(\theta) < \infty$  for all  $\theta > 0$ . Write

$$s_n(r) = m^{pn/r}(r) \mathbb{E}(W_n^{(r)})^{p/r} \quad (25)$$

for  $n \in \mathbb{N}_0$  and consider the WBP  $(\mathbf{V}, \mathbf{L}^r)$ . Since  $\theta \mapsto m^{-\theta}(r) \mathbb{E}(\sum_{|v|=1} L_v^{r\theta}) = m(r\theta)/m^\theta(r)$  has derivative

$$\frac{rm'(r\theta)}{m^\theta(r)} - \log m(r) \frac{m(r\theta)}{m^\theta(r)}$$

taking value  $r(\frac{m'(r)}{m(r)} - \log m^{1/r}(r)) = r^2 g'(r) < 0$  at  $\theta = 1$ , we infer (see [1, Theorem 1.3]) that  $W_n^{(r)}$  converges a.s. and in  $L_1$  to the random variable  $W^{(r)}$  which in turn entails (24) because, by (25) and an appeal to Jensen's

inequality and Fatou's lemma,

$$\begin{aligned}
\liminf_{n \rightarrow \infty} s_n^{1/n}(r) &= m^{p/r}(r) \liminf_{n \rightarrow \infty} \mathbb{E}^{1/n}(W_n^{(r)})^{p/r} \\
&= m^{p/r}(r) \liminf_{n \rightarrow \infty} \mathbb{P}^{1/n}(W^{(r)} > 0) \mathbb{E}^{1/n}\left((W_n^{(r)})^{p/r} \middle| W^{(r)} > 0\right) \\
&\geq m^{p/r}(r) \mathbb{E}\left(\liminf_{n \rightarrow \infty} (W_n^{(r)})^{p/r} \middle| W^{(r)} > 0\right) \\
&= m^{p/r}(r).
\end{aligned}$$

CASE B.  $p \leq \vartheta$  and  $r \in [1, \vartheta]$ . *Upper estimate*

The converse

$$\limsup_{n \rightarrow \infty} s_n^{1/n}(r) \leq m^{p/r}(r) \quad (26)$$

follows quite easily from (25), for  $\mathbb{E}(W_n^{(r)})^{p/r} \leq \mathbb{E}^{p/r} W_n^{(r)} = 1$  for each  $n \in \mathbb{N}_0$  in the case  $r \in [p, \vartheta]$  by Jensen's inequality, while in the case  $r \in [1, p]$  we have  $\sup_{n \geq 0} \mathbb{E}(W_n^{(r)})^{p/r} < \infty$  as a consequence of  $\mathbb{E}(W_1^{(r)})^{p/r} \leq m^{-p/r}(r) \mathbb{E} W_1^p < \infty$  and

$$E\left(\sum_{|v|=1} \left(\frac{L_v}{m(r)} W_1^{(r)}\right)^{p/r}\right) = \frac{m(p)}{m^{p/r}(r)} = e^{p(g(p)-g(r))} < 1$$

(apply Proposition 1.1 to  $W_n^{(r)}$ ,  $n \in \mathbb{N}_0$ ).

CASE C.  $p > \vartheta$  and  $r \in [1, q]$ . *Upper estimate.*

Notice that  $m(\vartheta) < \infty$ . As  $g(\vartheta) < g(p) < 0 = g(1)$ , there exists a unique  $1 < q < \vartheta$  such that  $g(q) = g(p)$ , i.e.  $m(q)^{1/q} = m(p)^{1/p}$ . Then, for  $r \in [1, q]$ , the previously given arguments are easily seen to carry over to the present situation thus showing (26).

CASE C.  $p > \vartheta$  and  $r \in [1, q]$ . *Lower estimate.*

By Jensen's inequality,

$$s_n(r) \geq \mathbb{E}^{p/q}(Z_n^{(r)})^{q/r}$$

But  $\mathbb{E}(Z_n^{(r)})^{q/r}$ , call it  $\tilde{s}_n(r)$ , is just the counterpart of  $s_n(r)$  for  $q < \vartheta$  instead of  $p$ . Therefore  $\tilde{s}_n^{1/n}(r) \rightarrow m^{q/r}(r)$  by what has been shown under Case B. It

thus follows that  $s_n^{1/n}(r)$  has also the required lower bound which completes the proof of

$$\lim_{n \rightarrow \infty} s_n^{1/n}(r) = m^{p/r}(r)$$

for all  $r \in [1, q)$ .

CASE D.  $p > \vartheta$  and  $r \in [q, 2]$ . *Upper estimate.*

Since  $s_n(q) = \inf_{\theta < q} s_n(\theta)$ , we obtain as a consequence of Case B that

$$\limsup_{n \rightarrow \infty} s_n^{1/n}(r) \leq \limsup_{n \rightarrow \infty} s_n^{1/n}(q) \leq \inf_{\theta \in [1, q]} m^{p/\theta}(\theta) = m^{p/q}(q) = m(p).$$

CASE D.  $p > \vartheta$  and  $r \in [q, 2]$ . *Lower estimate.*

The proof for Case C will now be completed by showing that

$$\liminf_{n \rightarrow \infty} s_n^{1/n}(2) \geq m(p) \quad (27)$$

(since  $s_n(r)$  is decreasing in  $r$ ) which is the most delicate part of the whole proof. Once again, possibly after a suitable truncation as described in (22), it is no loss of generality to assume that  $W_1 \leq K$  for some  $K \geq 1$ ,  $J \leq N$  for some  $N \in \mathbb{N}$  and  $m(\theta) < \infty$  for all  $\theta > 0$ . Notice also that, by subadditivity of  $x \mapsto x^{p/2}$ , we find

$$s_n(2) \leq m^n(p) \quad (28)$$

for all  $n \in \mathbb{N}_0$ .

Put  $\beta := 1 - (p/2) \in (0, 1)$ . Recall the notation introduced in Section 2 in connection with the size-biased probability measure  $\hat{\mathbb{P}}$ . We have

$$\begin{aligned} (Z_n^{(2)})^{p/2} &= \left( \Pi_n^2 + \sum_{k=1}^n \sum_{i \in \mathcal{I}_k} L_v^2 Z_{n-k}^{(2)}(\Xi_{k-1} i) \right)^{p/2} \\ &\leq K^p \left( \Pi_n^p + \sum_{k=1}^n \Pi_{k-1}^p (\Lambda_{n,k}^{(2)})^{p/2} \right) \quad \hat{\mathbb{P}}\text{-a.s.}, \end{aligned}$$

where, for  $1 \leq k \leq n$ ,

$$\Lambda_{n,k}^{(2)} := \sum_{i \in \mathcal{I}_k} Z_{n-k}^{(2)}(\Xi_{k-1} i).$$

Use Proposition 2.1(b), to see that conditioned upon  $\mathcal{G}_n$ , the  $Z_{n-k}^{(2)}(\Xi_{k-1}i), i \in \mathcal{I}_k$ , are i.i.d. under  $\widehat{\mathbb{P}}$  with the same distribution as  $Z_{n-k}^{(2)}$  under  $\mathbb{P}$ . By combining this with another subadditivity argument we obtain  $\widehat{\mathbb{P}}$ -a.s.

$$\widehat{\mathbb{E}}((\Lambda_{n,k}^{(2)})^{p/2}|\mathcal{G}_n) \leq \widehat{\mathbb{E}}\left(\sum_{i \in \mathcal{I}_1} (Z_{n-k}^{(2)}(\Xi_{k-1}i))^{p/2} \middle| \mathcal{G}_n\right) \leq |\mathcal{I}_1| \mathbb{E}(Z_{n-k}^{(2)})^{p/2}$$

for  $k = 1, \dots, n$ . As  $|\mathcal{I}_1| \leq N$  for some  $N \in \mathbb{N}$  by truncation we arrive at

$$\begin{aligned} \widehat{\mathbb{E}}((Z_n^{(2)})^{p/2}|\mathcal{G}_n) &\leq K^p N \left( \Pi_n^p + \sum_{k=1}^n \Pi_{k-1}^p \mathbb{E}(Z_{n-k}^{(2)})^{p/2} \right) \\ &= K^p N \sum_{k=0}^n \Pi_k^p s_{n-k}(2) \quad \widehat{\mathbb{P}}\text{-a.s.} \end{aligned}$$

Now use  $\mathbb{E}(Z_n^{(2)})^{p/2} = \widehat{\mathbb{E}}(\Pi_n(Z_n^{(2)})^{-\beta})$ , which follows from

$$\begin{aligned} \mathbb{E}(Z_n^{(2)})^{p/2} &= \widehat{\mathbb{E}}\left(\sum_{|v|=n} \frac{L_v^2}{W_n} (Z_n^{(2)})^{-\beta}\right) \\ &= \widehat{\mathbb{E}}\left(\sum_{|v|=n} \widehat{\mathbb{P}}(\Xi_n = v | \mathcal{F}_\infty) L_v (Z_n^{(2)})^{-\beta}\right) \\ &= \widehat{\mathbb{E}}\left(\sum_{|v|=n} \mathbf{1}_{\{\Xi_n = v\}} L_v (Z_n^{(2)})^{-\beta}\right) = \widehat{\mathbb{E}}\left(\Pi_n (Z_n^{(2)})^{-\beta}\right), \end{aligned}$$

to obtain by an appeal to Jensen's inequality for  $x \mapsto x^{(p-2)/p}$

$$\begin{aligned} \mathbb{E}(Z_n^{(2)})^{p/2} &= \widehat{\mathbb{E}}\left(\Pi_n \widehat{\mathbb{E}}((Z_n^{(2)})^{-\beta}|\mathcal{G}_n)\right) \\ &\geq \widehat{\mathbb{E}}\left(\frac{\Pi_n}{\widehat{\mathbb{E}}((Z_n^{(2)})^{p/2}|\mathcal{G}_n)^{(2-p)/p}}\right) \\ &\geq C \widehat{\mathbb{E}}\left(\frac{\Pi_n}{(\sum_{k=0}^n \Pi_k^p s_{n-k}(2))^{(2-p)/p}}\right) \end{aligned}$$

for some  $C > 0$ . Recall from Section 2 the definition of  $\widehat{\mathbb{P}}_p$  and that (see (11))

$$\widehat{\mathbb{P}}^{(p)}(B) = \frac{1}{m^n(p)} \widehat{\mathbb{E}}\left(\Pi_n^{p-1} \mathbf{1}_B\right)$$

for any  $B \in \mathcal{F}_n$ . The last expectation can be further estimated as

$$\begin{aligned}
& \widehat{\mathbb{E}} \left( \frac{\Pi_n}{\left( \sum_{k=0}^n s_{n-k}(2) \Pi_k^p \right)^{(2-p)/p}} \right) \\
&= m^n(p) \widehat{\mathbb{E}}^{(p)} \left( \frac{\Pi_n^{2-p}}{\left( \sum_{k=0}^n s_{n-k}(2) \Pi_k^p \right)^{(2-p)/p}} \right) \\
&= m^n(p) \widehat{\mathbb{E}}^{(p)} \left( \frac{1}{\left( \sum_{k=0}^n s_{n-k}(2) (\Pi_n / \Pi_k)^p \right)^{(2-p)/p}} \right) \\
&= m^n(p) \widehat{\mathbb{E}}^{(p)} \left( \frac{1}{\left( \sum_{k=0}^n s_k(2) \Pi_k^p \right)^{(2-p)/p}} \right) \\
&\geq m^n(p) \widehat{\mathbb{E}}^{(p)} \left( \frac{1}{\left( \sum_{k=0}^n m^k(p) \Pi_k^p \right)^{(2-p)/p}} \right) \\
&\geq m^n(p) \widehat{\mathbb{E}}^{(p)} \left( \frac{1}{\left( \sum_{k \geq 0} (\Pi_k^*)^p \right)^{(2-p)/p}} \right)
\end{aligned}$$

where (28) has been utilized for the penultimate inequality and where  $\Pi_k^* := \Pi_k / m^{k/p}(p)$  for  $k \in \mathbb{N}_0$ . Since  $\widehat{\mathbb{E}}^{(p)} \Pi_1^\theta = \frac{m(p+\theta)}{m(p)}$  for all  $\theta \in \mathbb{R}$  we find

$$\widehat{\mathbb{E}}^{(p)} \log \Pi_1^* = \frac{m'(p)}{m(p)} - \frac{1}{p} \log m(p) = \frac{h(p)}{p}.$$

Now use  $p > \vartheta$  to infer  $\widehat{\mathbb{E}}^{(p)} \ln \Pi_1^* > 0$  and thereupon that

$$1 < \Sigma := \sum_{k \geq 0} (\Pi_k^*)^{-p} < \infty \quad \widehat{\mathbb{P}}_p\text{-a.s.},$$

in particular  $\nu := \widehat{\mathbb{E}}^{(p)} \Sigma^{-(2-p)/p} \in (0, 1)$ . We finally arrive at

$$m^n(p) \geq s_n(2) = \mathbb{E} Z_n^{p/2} \geq \nu m^n(p) \quad (29)$$

for all  $n \in \mathbb{N}_0$  which clearly implies the desired assertion (27). The proof of Proposition 3.4 is thus complete.  $\square$

The next lemma is needed for the proof of Theorem 1.4 and examines the asymptotic behaviour of  $\mathbb{E}W_n^p$  when  $\mathbb{E}W_1^p < \infty$  and  $m(p) \geq 1$ . It may also be viewed as a useful complement to Proposition 1.1. Let us mention that we have not tried to obtain the best possible estimates. Actually, for our purposes only factors of exponential growth matter. Hence, we content ourselves with quite crude estimates when dealing with factors of subexponential growth.

**Lemma 3.5.** *Let  $p > 1$  and  $\mathbb{E}W_1^p < \infty$ .*

- (a) *If  $p \in (1, 2]$ , then  $\mathbb{E}W_n^p = O(n)$  if  $m(p) = 1$  and  $\mathbb{E}W_n^p = O(m^n(p))$  if  $m(p) > 1$ .*
- (b) *If  $p > 2$ , then  $\mathbb{E}W_n^p = O(n^{p-1})$  if  $m(p) = 1$  and  $\mathbb{E}W_n^p = O(n^{b(p-1)}m^n(p))$  for  $p \in (b+1, b+2]$ ,  $b \in \mathbb{N}$ , if  $m(p) > 1$ .*

We note in advance that the lemma will later be applied to the martingale  $\{W_n^{(2)} : n \in \mathbb{N}_0\}$  rather than to  $\{W_n : n \in \mathbb{N}_0\}$ .

*Proof.* (a) Use Proposition 2.1(e) with  $f(x) = x^{p-1}$  to infer

$$\begin{aligned} \mathbb{E}W_n^p &\leq \widehat{\mathbb{E}} \left( \sum_{k=0}^{n-1} M_1 M_2 \cdots M_k Q_{k+1} \right)^{p-1} \\ &\leq \widehat{\mathbb{E}} \left( \sum_{k=0}^{n-1} (M_1 M_2 \cdots M_k)^{p-1} Q_{k+1}^{p-1} \right) = \widehat{\mathbb{E}} Q^{p-1} \sum_{k=0}^{n-1} \widehat{\mathbb{E}}^k M^{p-1}, \end{aligned}$$

where in the next to last inequality the subadditivity of  $f$  has been utilized and  $(M, Q)$  denotes a generic copy of the  $(M_n, Q_n)$ . In view of Proposition 2.1,  $\widehat{\mathbb{E}} Q^{p-1} = \mathbb{E}W_1^p$  and  $\widehat{\mathbb{E}} M^{p-1} = m(p)$ , and the result follows.

(b) Put  $\varphi_n(s) := \mathbb{E}e^{-sW_n}$  for  $n \in \mathbb{N}_0$ . Then

$$\varphi_n(s) = \mathbb{E} \prod_{|v|=1} \varphi_{n-1}(sL_v), \quad n \in \mathbb{N}.$$

Differentiating this equality yields

$$\varphi'_n(s) = \sum_{|v|=1} \varphi'_{n-1}(sL_v) L_v \prod_{u \neq v} \varphi_{n-1}(sL_u), \quad n \in \mathbb{N}. \quad (30)$$

It is known and readily checked that  $-\varphi'_n(s)$  is the Laplace transform of  $W_n$  under the size-biased measure  $\widehat{\mathbb{P}}$ . Let  $V_n$  be a random variable with  $\mathbb{P}(V_n \in \cdot) = \widehat{\mathbb{P}}(W_n \in \cdot)$  (the use of  $V_n$  is for our convenience and allows us to do all subsequent calculations under  $\mathbb{P}$  only). Then (30) is equivalent to the distributional identity

$$V_n \stackrel{d}{=} MV_{n-1} + T_n, \quad n \in \mathbb{N}, \quad (31)$$

where  $(M, T_n)$  is a random vector independent of  $V_{n-1}$  and with distribution

$$\begin{aligned} \mathbb{P}\{(M, T_n) \in B\} &= \mathbb{E} \left( \sum_{|u|=1} L_u \mathbf{1}_B \left( L_u, \sum_{v \neq u} L_v W_{n-1}(v) \right) \right) \\ &= \widehat{\mathbb{P}} \left\{ \left( \Pi_1, \sum_{v \in \mathcal{I}_1} L_v W_{n-1}(v) \right) \in B \right\}, \quad n \in \mathbb{N}, \end{aligned} \quad (32)$$

where  $B \subset \mathbb{R}^2$  is any Borel set. An application of Minkowski's inequality in  $L_{p-1}$  yields

$$\|V_n\|_{p-1} \leq \|M\|_{p-1} \|V_{n-1}\|_{p-1} + \|T_n\|_{p-1}. \quad (33)$$

Arguing in the same way as in the proof of Lemma 4.2 in [13] one finds that

$$\|T_n\|_{p-1} \leq \|W_{n-1}\|_{p-1} (\mathbb{E} W_1^p)^{1/(p-1)}. \quad (34)$$

For the remaining discussion we distinguish between two cases:

CASE 1.  $m(p) = 1$

Note that  $\|M\|_{p-1}^{p-1} = \widehat{\mathbb{E}} \Pi_1^{p-1} = m(p) = 1$  and that  $m(p-1) < 1$  (by log-convexity of  $m$ ). Hence  $\sup_{n \geq 0} \|W_n\|_{p-1} < \infty$  by Proposition 1.1, and we obtain from (33) that  $\|V_n\|_{p-1} = O(n)$  or, equivalently,

$$\mathbb{E} W_n^p = \mathbb{E} V_n^{p-1} = O(n^{p-1}).$$

CASE 2.  $m(p) > 1$

Assume first that  $p \in (2, 3]$ . Then we conclude from (34) and the already established part of the lemma that  $\|T_n\|_{p-1} = O(m^{n/(p-1)}(p))$ , regardless of the (finite) value of  $m(p-1)$ . By (32),  $\|M\|_{p-1} = m^{1/(p-1)}(p)$  whence we conclude from (33) that  $\|V_n\|_{p-1} = O(nm^{n/(p-1)}(p))$  or, equivalently, that

$$\mathbb{E} W_n^p = \mathbb{E} V_n^{p-1} = O(n^{p-1} m^n(p)).$$

The subsequent proof proceeds by induction over  $b$ . Suppose that we have already verified that  $\mathbb{E}W_n^p = O(n^{b(p-1)}m^n(p))$  when  $p \in (b+1, b+2]$ . In order for proving  $\mathbb{E}W_n^p = O(n^{(b+1)(p-1)}m^n(p))$  when  $p \in (b+2, b+3]$  it suffices to note that  $\|T_n\|_{p-1} = O(n^b m^{n/(p-1)}(p))$  and that any solution to the recursive inequality

$$c_n \leq dc_{n-1} + O(n^b d^n), \quad n \in \mathbb{N}, \quad c_0 = 1,$$

with  $d > 1$  satisfies  $c_n = O(n^{b+1}d^n)$ . This completes the proof.  $\square$

We mention in passing that the distributional identity (31), obtained above with the help of Laplace transforms (see (30)), may also be derived by a probabilistic argument using the results stated in Section 2 on size biasing and spinal trees. However, we refrain from supplying further details.

As we will make multiple use of the following version of Burkholder's inequality (see [8, Theorem 1 on p. 396]), it is stated here for ease of reference.

**Lemma 3.6.** *Let  $p > 1$  and  $\{Z_n : n \in \mathbb{N}\}$  be a martingale with  $Z_0 = 0$  and a.s. limit  $A$ . Then  $\mathbb{E}|Z|^p < \infty$  if and only if  $\mathbb{E}(\sum_{n \geq 0} (Z_{n+1} - Z_n)^2)^{p/2} < \infty$ . If one of these holds then*

$$c_p \left\| \left( \sum_{n \geq 0} (Z_{n+1} - Z_n)^2 \right)^{1/2} \right\|_p \leq \|Z\|_p \leq C_p \left\| \left( \sum_{n \geq 0} (Z_{n+1} - Z_n)^2 \right)^{1/2} \right\|_p,$$

where  $c_p := (p-1)/(18p^{3/2})$  and  $C_p := 18p^{3/2}/(p-1)^{1/2}$ .

## 4 Proofs of Theorem 1.2 and 1.4

Before proceeding with the proof of the main results, put  $\mu_p := \mathbb{E}|W_1 - 1|^p$ ,  $R := \sum_{n \geq 0} e^{2an} (W_{n+1} - W_n)^2$  and recall that  $W_n(v)$  denotes the copy of  $W_n$  pertaining to the subtree  $v\mathbf{V}$  rooted at  $v$ . Then

$$W_{n+1} - W_n = \sum_{|v|=n} L_v(W_1(v) - 1) \tag{35}$$

for  $n \in \mathbb{N}_0$ . Let us also stipulate hereafter that  $C \in (0, \infty)$  denotes a generic constant which may differ from line to line.

*Proof of Theorem 1.2.* In view of Burkholder's inequality (Lemma 3.6) it is clear that  $L_p$ -convergence of  $\hat{A} = \sum_{n \geq 0} e^{an}(W_{n+1} - W_n)$  holds true if, and only if,  $R$  exists in  $L_{p/2}$ . Suppose the latter be true and recall that  $\gamma = \inf\{m^{1/r}(r) : r \in [1, 2]\}$ . Then, by a double use of Jensen's inequality in combination with Burkholder's inequality,

$$\begin{aligned}
\mathbb{E}R^{p/2} &= \mathbb{E} \left( \sum_{n \geq 0} \left( \sum_{|v|=n} e^{an} L_v(W_1(v) - 1) \right)^2 \right)^{p/2} \\
&\geq N^{p/2} \mathbb{E} \left( \frac{1}{N} \sum_{n=0}^{N-1} \left( \sum_{|v|=n} e^{an} L_v(W_1(v) - 1) \right)^2 \right)^{p/2} \\
&\geq \frac{1}{N^{1-p/2}} \sum_{n=0}^{N-1} \mathbb{E} \left| \sum_{|v|=n} e^{an} L_v(W_1(v) - 1) \right|^p \\
&\geq \frac{C}{N^{1-p/2}} \sum_{n=0}^{N-1} \mathbb{E} \left( \sum_{|v|=n} e^{2an} L_v^2(W_1(v) - 1)^2 \right)^{p/2} \\
&= \frac{1}{N^{1-p/2}} \sum_{n=0}^{N-1} e^{pan} \mathbb{E} \left( Z_n^{(2)} \sum_{|v|=n} \frac{L_v^2}{Z_n^{(2)}} (W_1(v) - 1)^2 \right)^{p/2} \\
&\geq \frac{C\mu_p}{N^{1-p/2}} \sum_{n=0}^{N-1} e^{pan} \mathbb{E} (Z_n^{(2)})^{p/2} \\
&= \frac{C\mu_p}{N^{1-p/2}} \sum_{n=0}^{N-1} e^{pan} s_n(2).
\end{aligned}$$

This proves necessity of  $\mu_p < \infty$  and  $\lim_{n \rightarrow \infty} e^{pa} s_n^{1/n}(2) \leq 1$ . But the last limit equals either  $e^{pa}\gamma^p$  (if  $p \leq \vartheta$ ) or  $e^{pa}m(p)$  by an appeal to Proposition 3.4. Therefore we have proved necessity of  $e^a \inf_{r \in [p, 2]} m^{1/r}(r) \leq 1$  as claimed.

Now suppose that  $\mu_r < \infty$  and  $e^a m^{1/r}(r) < 1$  for some  $r \in [p, 2]$ . By an appeal to Burkholder's inequality in combination with the subadditivity of

$x \mapsto x^{p/2}$  and  $x \mapsto x^{r/2}$ , we infer

$$\begin{aligned} \mathbb{E}R^{p/2} &\leq \sum_{n \geq 0} e^{pan} \mathbb{E}|W_{n+1} - W_n|^p \\ &\leq C \sum_{n \geq 0} e^{pan} \mathbb{E} \left( \sum_{|v|=n} L_v^2(W_1(v) - 1)^2 \right)^{p/2} \\ &\leq C \sum_{n \geq 0} e^{pan} \mathbb{E} \left( \sum_{|v|=n} L_v^r |W_1(v) - 1|^r \right)^{p/r}. \end{aligned}$$

Use Jensen's inequality to see that

$$\mathbb{E} \left( \sum_{|v|=n} L_v^r |W_1(v) - 1|^r \right)^{p/r} \leq \mu_r^{p/r} m^{p/r}(r)$$

and thus

$$\mathbb{E}R^{p/2} \leq C \mu_r^{p/r} \sum_{n \geq 0} e^{pan} m^{p/r}(r) < \infty.$$

This completes the proof of Theorem 1.2.  $\square$

**Remark 4.1.** If  $p > \vartheta$ , i.e.  $m^{1/\vartheta}(\vartheta) < m^{1/p}(p) = \min_{r \in [p, 2]} m^{1/r}(r)$ , the proof of Proposition 3.4 (see (29)) has actually shown that  $\nu m^n(p) \leq s_n(2) \leq m^n(p)$  for all  $n \in \mathbb{N}_0$  and some  $\nu \in (0, 1)$ . In this case we hence obtain

$$\mathbb{E}R^{p/2} \geq \frac{\mu_p}{N^{1-p/2}} \sum_{n=0}^{N-1} e^{pan} s_n(2) \geq \frac{\nu \mu_p}{N^{1-p/2}} \sum_{n=0}^{N-1} e^{pan} m^n(p)$$

and thereby conclude that the  $L_p$ -convergence of  $\hat{A}$  or, equivalently,  $\mathbb{E}R^{p/2} < \infty$  can only hold true if  $e^a m^{1/p}(p) < 1$ . In the case where the function  $r \mapsto m^{1/r}(r)$  attains its minimum at some  $\theta \geq 2$ , we arrive at a similar conclusion, because then  $\lim_{n \rightarrow \infty} m^{-np/2}(2) s_n(2)$  exists and is positive by (21). This confirms our assertions stated in Remark 1.3.

*Proof of Theorem 1.4.* We show first necessity of condition (7) and thus assume that the series  $A$  in (4) converges in  $L_p$ . By Lemma 3.1 and Remark 3.2(a), the same holds true for  $\hat{A}$  and by an appeal to Lemma 3.6

$$\mathbb{E}R^{p/2} < \infty \tag{36}$$

As  $p \geq 2$ , the function  $x \mapsto x^{p/2}$  is superadditive and thus

$$\sum_{n \geq 0} e^{pan} \mathbb{E}|W_{n+1} - W_n|^p \leq \mathbb{E}R^{p/2} < \infty. \quad (37)$$

It is clear from (35) that  $W_{n+1} - W_n$  is the a.s. limit of a martingale (see, for example, [2, Section 3] for more details). Consequently, by another appeal to Lemma 3.6 and the afore-mentioned superadditivity,

$$\begin{aligned} \mathbb{E}|W_{n+1} - W_n|^p &\geq C \mathbb{E} \left( \sum_{|v|=n} L_v^2(W_1(v) - 1)^2 \right)^{p/2} \\ &\geq C \mathbb{E} \left( \sum_{|v|=n} L_v^p |W_1(v) - 1|^p \right) \\ &= C \mu_p m^n(p) \end{aligned}$$

for  $n \in \mathbb{N}_0$ . This inequality together with (37) implies the necessity of  $\mathbb{E}W_1^p < \infty$  and  $e^{ap}m(p) < 1$  for the  $L_p$ -convergence of  $A$ . Moreover, by Jensen's inequality,

$$\begin{aligned} \mathbb{E} \left( \sum_{|v|=n} L_v^2(W_1(v) - 1)^2 \right)^{p/2} &\geq \left( \mathbb{E} \left( \sum_{|v|=n} L_v^2(W_1(v) - 1)^2 \right) \right)^{p/2} \\ &= \mu_2^{p/2} m(2)^n \end{aligned}$$

which together with (37) finally gives the asserted necessity of  $e^a m^{1/2}(2) < 1$ .

Let us now turn to the sufficiency of conditions (7). By Lemma 3.1 and Remark 3.2(a) it suffices to verify  $L_p$ -convergence of  $\hat{A}$ . By combining (35), another use of Lemma 3.6, the convexity of  $x \mapsto x^{p/2}$  and a conditioning with respect to  $\mathcal{F}_n$ , we infer

$$\begin{aligned} \mathbb{E}|W_{n+1} - W_n|^p &\leq C \mathbb{E} \left( \sum_{|v|=n} L_v^2(W_1(v) - 1)^2 \right)^{p/2} \\ &\leq C \mathbb{E} \left( Z_n^{(2)} \sum_{|v|=n} \frac{L_v^2}{Z_n^{(2)}} (W_1(v) - 1)^2 \right)^{p/2} \\ &\leq C \mu_p \mathbb{E}(Z_n^{(2)})^{p/2} \end{aligned}$$

whence it is enough to verify

$$e^{pan} \mathbb{E}(Z_n^{(2)})^{p/2} = O(q^n) \quad \text{for some } q \in (0, 1), \quad (38)$$

hereafter. Indeed, it then follows that

$$e^{2an} \mathbb{E}^{2/p} |W_{n+1} - W_n|^p = O(q^{2n/p}). \quad (39)$$

and thereby with the help of Minkowski's inequality in  $L_{p/2}$  and once more Lemma 3.6

$$\mathbb{E}|\hat{A}|^p \leq C \mathbb{E}R^{p/2} \leq C \left( \sum_{n \geq 0} e^{2an} \mathbb{E}^{2/p} |W_{n+1} - W_n|^p \right)^{p/2} < \infty.$$

So let us prove (38) for the case  $p > 2$ , for it trivially holds with  $q = e^{2a}m(2)$  in the case  $p = 2$ . Notice that  $\mathbb{E}(W_1^{(2)})^{p/2} \leq m^{-p/2}(2)\mathbb{E}W_1^p < \infty$ . For the remaining discussion we distinguish two cases:

CASE 1.  $m(p) < m^{p/2}(2)$ .

Then the second condition in (7) reads  $e^am^{1/2}(2) < 1$ . By Proposition 1.1 applied to  $W_n^{(2)}$  and  $p/2$  instead of  $W_n$  and  $p$ , we obtain  $\sup_{n \geq 0} \mathbb{E}(W_n^{(2)})^{p/2} < \infty$  and this ensures validity of (38) with  $q = e^{ap}m^{p/2}(2)$ .

CASE 2.  $m(p) \geq m^{p/2}(2)$ .

Then the second condition in (7) takes the form  $e^am^{1/p}(p) < 1$ . Lemma 3.5 applied to  $W_n^{(2)}$  and  $p/2$  instead of  $W_n$  and  $p$  provides us with  $\mathbb{E}(W_n^{(2)})^{p/2} = O(n^c m^n(p) m(2)^{-pn/2})$  for  $c > 0$ . Consequently,  $\mathbb{E}(Z_n^{(2)})^{p/2} = O(n^c m^n(p))$  which proves validity of (38) with  $q = \delta e^{pa}m(p)$  for some  $\delta > 1$  sufficiently close to 1. The proof is herewith complete.  $\square$

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